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1990 J. Phys. A: Math. Gen. 23 2879

(http://iopscience.iop.org/0305-4470/23/13/025)

J. Phys. A: Math. Gen. 23 (1990) 2879–2896. Printed in the UK

Electromagnetic wave propagation in turbulent and nonlinear plasmas

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Received 23 January 1990, in final form 28 March 1990

Abstract. The propagation of an electromagnetic wave in a time stationary turbulent and nonlinear plasma is studied with a view to ascertaining the statistical moments of the wavefield. We find that the functional method which was used by Hopf in his study of ordinary turbulence is a powerful one also for turbulent and nonlinear media. A functional differential equation is derived for the moments of electromagnetic waves propagating in an isotropic plasma in which the dielectric constant undergoes statistical fluctuations. Using the Markov and small-angle forward-scattering approximations, we find a hierarchy of coupled partial differential equations for the moments containing different wave numbers. An approximate perturbation method is devised for decoupling and solving the hierarchy to any desired order. We draw attention to the similarity of the closure problem of the moment equations to that in turbulence generally and in statistical mechanics. Possible applications are discussed.

1. Introduction

Electromagnetic wave propagation in miscellaneous random media has attracted much attention [2–15]. This is due not only to the possible practical benefits but also to the scientific richness and value of this problem. There have been several studies of wave propagation in nonlinear media containing random inhomogeneities [12–15]. For a linear medium containing random inhomogeneities, the effects of inhomogeneities can be taken into account formally up to an arbitrary order of accuracy by various methods. These include the geometrical optics method [3], the method of smoothing perturbation [2–5], the renormalisation-diagram technique [6] and the method of successive scatters [5-7]. Fluctuations in the electromagnetic wave are seen as resulting from interference between randomly modulated wavefields caused by random fluctuations in the medium. In a nonlinear turbulent plasma random inhomogeneities and random electromagnetic characteristics are produced by the turbulent fluctuations. The nonlinear response of the turbulent plasma must, therefore, be taken into account, and this makes the problem considerably more complicated. Phenomena requiring this fuller treatment include self-induced transparency, self-focusing, self-modulation and the generation and collapse of solitary waves [16–18].

In studies of wave propagation in turbulent and nonlinear plasmas, the statistical characteristics of the wave are often needed. In order to obtain these we begin

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0305-4470/90/132879+18$03.50

by defining the moments of the electromagnetic wavefield and the characteristics of their statistical properties. The electric field of a high frequency monochromatic electromagnetic wave propagating in a turbulent and nonlinear plasma can be expressed as

$$\boldsymbol{E} = \boldsymbol{E}_{\boldsymbol{k},\omega} \exp[-\mathrm{i}(\omega t - \boldsymbol{k}\boldsymbol{r})]. \tag{1}$$

If we assume that the characteristic scale length of the inhomogeneities in the medium is much larger than the wavelength, and the inhomogeneities are weak, the scattering produced by the inhomogeneities will always be at a small angle to the direction of propagation [2]. Under this small-angle approximation, we can use a scalar wave equation to describe the propagation. We will also assume that the process is a Markov process, as been found in many experiments [3]. Under these conditions, the m-nth moment of the wavefield is the correlation function of the wavefield, which can be defined by

$$\Gamma_{mn} = \left\langle \prod_{j=1}^{m} \boldsymbol{E}_j(\boldsymbol{r}_j, t) \prod_{l=1}^{n} \boldsymbol{E}_l^*(\boldsymbol{r}_l', t') \right\rangle \tag{2}$$

where $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the radius vectors, $\langle\cdot\rangle$ denotes ensemble average, j and l are integers.

When the characteristics of the turbulent plasma are time invariant, the m-nth moment of the field is [19]

$$\begin{aligned}\left\langle \prod_{j=1}^{m} \boldsymbol{E}_j(\boldsymbol{r}_j, t) \prod_{l=1}^{n} \boldsymbol{E}_l^*(\boldsymbol{r}_l', t') \right\rangle &= \int \left\langle \prod_{j=1}^{m} \boldsymbol{E}(\boldsymbol{k}_j, \omega_j) \prod_{l=1}^{n} \boldsymbol{E}^*(\boldsymbol{k}_j', \omega_l') \right\rangle \\ &\quad \times \exp\left[\mathrm{i}\left(\sum_{j=1}^{m} \boldsymbol{k}_j \boldsymbol{r}_j - \sum_{l=1}^{n} \boldsymbol{k}_l' \boldsymbol{r}_l'\right) - \mathrm{i}\left(\sum_{j=1}^{m} \omega_j - \sum_{l=1}^{n} \omega_l'\right) t\right] \\ &\quad \times \prod_{j=1}^{m} \mathrm{d}\omega_j \mathrm{d}\boldsymbol{k}_j \prod_{l=1}^{n} \mathrm{d}\omega_l' \mathrm{d}\boldsymbol{k}_l'.\end{aligned} \tag{3}$$

For the plasma to be time invariant, the coefficient of t must be zero, i.e.

$$\sum_{j=1}^{m} \omega_j - \sum_{l=1}^{n} \omega_l' = 0. \tag{4}$$

The old method [15] of deriving the equations of the moment of the wavefield, is to multiply the wave equation by the wavefield and its conjugates, and then take the average. In order to treat the average terms containing fluctuating physical quantities as well as the wavefield, it is necessary to assume that the fluctuating physical quantity has a Gaussian distribution and then use the Donsker–Frutsu–Novikov formula [2–5, 15]. However, random fluctuations of physical quantities in a turbulent plasma are not necessarily Gaussian. For dealing with the general case, therefore, we shall adopt a different approach—that of the characteristic functional method initiated by Hopf to study turbulence in fluids [1].

In section 2 we shall derive the functional differential equation and hierachy of coupled partial differential equations for moments of the wavefield with different wavenumbers in an isotropic, time-stationary plasma with random inhomogeneities and weak nonlinearity. In section 3, we shall use a perturbation method to obtain an approximate solution when the nonlinearity is cubic. In last section we discuss a closure problem encountered in section 3, and the significance and possible applications of our results.

2. The hierarchy of coupled partial differential equations of wave moments in a turbulent and nonlinear plasma

We mentioned in the previous section that a scalar equation is adequate for describing the propagation of high frequency electromagnetic waves in a turbulent and nonlinear plasma. We now derive the equations for the moments of the wavefield. The hierarchy of coupled partial differential equations for a single wavenumber in a random medium with Gaussian statistical characteristics in addition to cubic nonlinearity were derived by using the method of averaging [15]. However, as we have seen, a wave propagating in a turbulent and nonlinear plasma becomes modulated and develops a finite range of wavenumbers. Neither are the statistical characteristics of turbulent media necessarily Gaussian. We shall need therefore, to use a different approach, in which the functional method is employed to derive the complete set of the moment equations for the wavefield with different position and different wavenumbers. No prescriptive limitation is imposed on the statistical characteristics.

2.1. The stochastic nonlinear partial differential equation for the wave

When a high frequency electromagnetic wave is propagating in a turbulent plasma, the electric field of the wave obeys the following stochastic nonlinear partial differential equation [7]

$$\nabla^2 E(\boldsymbol{r},\omega) + (\omega/c)^2 \epsilon(\boldsymbol{r},\omega,|E|)E(\boldsymbol{r},\omega) = 0 \tag{5}$$

where $\boldsymbol{r}$ is the three-dimensional radius vector, ω the wave angular frequency, and ϵ the relative permitivity. We shall consider ϵ to be composed of a mean value,ϵ_0; the local deviation averaged over the characteristic scale length of the fluctuations, $\epsilon_1\epsilon_0$; the local irregularity,$\epsilon_2\epsilon_0$ and a nonlinear component $\epsilon_3\epsilon_0$ where

$$k^2(\omega) = (\omega/c)^2 \epsilon_0(\omega) \tag{6}$$

$$\epsilon_1(\boldsymbol{r},\omega) = \frac{|\langle\epsilon_1(\boldsymbol{r},\omega)\rangle - \epsilon_0(\omega)|}{\epsilon_0(\omega)} \tag{7}$$

$$\epsilon_2(\boldsymbol{r},\omega) = \frac{\delta\epsilon_1(\boldsymbol{r},\omega)}{\epsilon_0(\omega)} \tag{8}$$

$$\epsilon_3(\omega,|E|) = \frac{\epsilon_{\rm n}(\omega,|E|)}{\epsilon_0(\omega)} \tag{9}$$

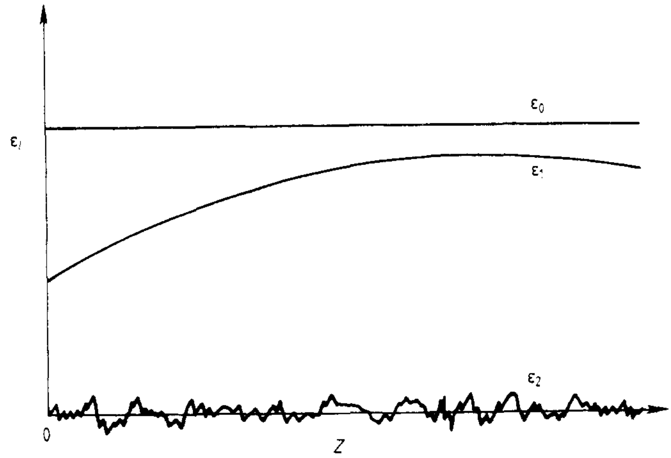

Figure 1. Sketch of the linear components as a function of z, of the relative permeability of a turbulent and nonlinear plasma. ϵ_0, ϵ_1 and ϵ_2 are average, spatial deviation and random part of the linear components of relative permeability.

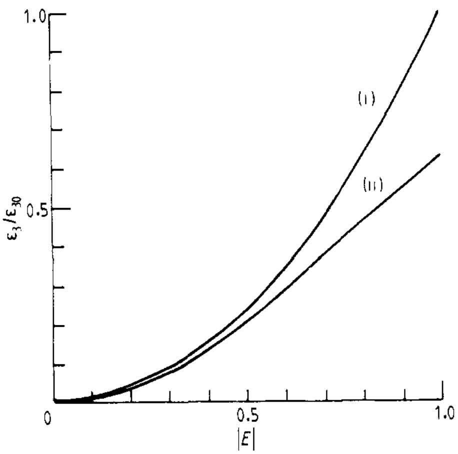

Figure 2. Sketch of the nonlinear component as a function of $|E|$, of the relative permeability of a turbulent and nonlinear plasma. The curve (i) represents the quadratic dependence of ϵ_3 on $|E|$ specified in (51*b*); and (ii) the exponential dependence given in (14).

in which l denotes the linear part, and n the nonlinear part. The components of the relative permitivity of a turbulent nonlinear plasma are shown schematically in figures 1 and 2.

Using (6)–(9), equation (5) becomes

$$\nabla^2 E(\boldsymbol{r},\omega) + k^2(\omega)[1 + \epsilon_1(\boldsymbol{r},\omega) + \epsilon_2(\boldsymbol{r},\omega)]E(\boldsymbol{r},\omega) + k^2(\omega)\epsilon_3(\omega,|E|)E(\boldsymbol{r},\omega) = 0. \quad (10)$$

This is a nonlinear partial differential equation with stochastic coefficients. Clearly, ϵ_2 represents fluctuations in the plasma, ϵ_1 accounts for the deterministic spatial variation, and ϵ_3 expresses the nonlinearity. Since the inhomogeneity is weak, and the small angle forward scattering approximation is assumed valid, see figure 3, we can

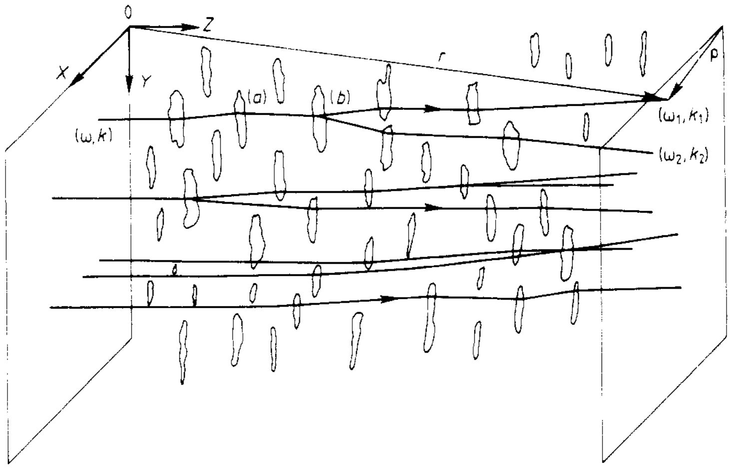

Figure 3. Schematic of wave propagation in a turbulent and nonlinear plasma when the small-angle forward scattering approximation is valid. (a) is an example of linear scattering, (b) of nonlinear scattering and scattering centres are indicated by irregular closed surfaces. Note that the scattering centres are more extensive in the x and y directions than along z, as is required by the forward scattering approximation to be valid.

use the paraxial approximation. Let

$$E(\boldsymbol{r},\omega) = u(\boldsymbol{\rho}, z, \omega)\exp(\mathrm{i}kz) \tag{11}$$

where $\boldsymbol{r} = (\boldsymbol{\rho}, z)$, $\boldsymbol{\rho}$ is the two-dimensional radius vector. Putting this into equation (10) and assuming that $u(\boldsymbol{\rho}, z, \omega)$ does not change significantly along z we have the following nonlinear stochastic differential equation for u:

$$\frac{\mathrm{i}}{k}\frac{\partial}{\partial z}u = -\frac{1}{2k^2}\nabla_\perp^2 u - \tfrac{1}{2}\epsilon_1 u - \tfrac{1}{2}\epsilon_2 u - \tfrac{1}{2}\epsilon_3(\omega, |E|)u \tag{12a}$$

where $\nabla_\perp^2$ is the two-dimensional Laplacian operator. The equation can also be written as

$$\frac{\partial}{\partial z}u = \frac{\mathrm{i}}{2k}\Delta_\perp^2 u + \frac{\mathrm{i}}{2}k\epsilon_2(\boldsymbol{r},\omega)u + \frac{\mathrm{i}}{2}k\epsilon_3(\omega, |E|)u \tag{12b}$$

where the operator $\Delta_\perp^2$ is defined as

$$\Delta_\perp^2 = \nabla_\perp^2 + k^2\epsilon_1. \tag{13}$$

The nonlinearity of the dielectric constant ϵ_3 is dependent on the strength of the electric field. If it has a quadratic dependance, equation (12b) becomes a cubic Schrödinger equation. However if ϵ_3 is of more complex form e.g.

$$\epsilon_3(\omega, |E|) = \epsilon_{30}[1 - \exp(-\alpha|E|^2)] \tag{14}$$

where ϵ_{30} is a small constant and α is a normalising constant, then equation (12b) becomes an exponential nonlinear Schrödinger equation. We shall discuss the problems of different types of nonlinearity below.

2.2. The functional differential equation for the characteristic functional of the wave

The characteristic functional of the wavefield can be defined as

$$\Psi(z,\nu,\nu^*) = \left\langle \exp\left(\mathrm{i} \int [u(\rho,z,k)\nu(\rho,k) + u^*(\rho,z,k)\nu^*(\rho,k)] \mathrm{d}\rho \, \mathrm{d}k \right) \right\rangle \equiv \langle \exp(\mathrm{i}R) \rangle \tag{15}$$

where * denotes complex conjugation, and the range of integration is over all allowed values of ρ and k. The ν and ν^* here are independent functions of ρ and k. By differentiating (15) with respect to z, we obtain

$$\begin{aligned}\frac{\delta\Psi}{\delta z} = \Bigg\langle \exp(\mathrm{i}R) \int \Bigg\{ &\nu \left[\frac{\mathrm{i}}{2k}\Delta_\perp^2 u + \frac{\mathrm{i}}{2}k\epsilon_2(r,\omega)u + \frac{\mathrm{i}}{2}k\epsilon_3(\omega,|E|)u \right] \\ &-\nu^* \left[\frac{\mathrm{i}}{2k}\Delta_\perp^2 u^* + \frac{\mathrm{i}}{2}k\epsilon_2(r,\omega)u^* + \frac{\mathrm{i}}{2}k\epsilon_3 u^* \right] \Bigg\} \mathrm{d}\rho \, \mathrm{d}k \Bigg\rangle .\end{aligned} \tag{16}$$

From (15), we have also the following functional derivatives:

$$\mathrm{i}\langle u \exp(\mathrm{i}R) \rangle = \frac{\delta\Psi}{\delta\nu} \tag{17a}$$

and

$$\mathrm{i}\langle u^* \exp(\mathrm{i}R) \rangle = \frac{\delta\Psi}{\delta\nu^*}. \tag{17b}$$

Operating with $\Delta_\perp^2$ on (17*a*) and (17*b*), we have

$$\langle \Delta_\perp^2 u \exp(\mathrm{i}R) \rangle = \frac{1}{\mathrm{i}}\Delta_\perp^2 \frac{\delta\Psi}{\delta\nu} \tag{18a}$$

and

$$\langle \Delta_\perp^2 u^* \exp(\mathrm{i}R) \rangle = \frac{1}{\mathrm{i}}\Delta_\perp^2 \frac{\delta\Psi}{\delta\nu^*}. \tag{18b}$$

To deal with other terms of equation (16), we define following functional

$$g(\nu,\nu^*,z,\rho,k) \equiv \langle \exp(\mathrm{i}R)\beta(z,\rho) \rangle \tag{19}$$

where we let

$$\tfrac{1}{2}\mathrm{i}k\epsilon_2(r,\omega) + \tfrac{1}{2}\mathrm{i}k\epsilon_3(\omega,|E|) \equiv \beta(z,\rho,k). \tag{20}$$

By expanding $\exp(\mathrm{i}R)$, we can write (19) as follows

$$\begin{aligned} &g(\nu,\nu^*,z,\rho,k) \\ &\quad = \sum_{m=0}^{\infty} \frac{\mathrm{i}^m}{m!} \left\langle \left[\int (u_1\nu_1 + u_1^*\nu_1^*)\mathrm{d}\sigma_1 \right] \cdots \left[\int (u_m\nu_m + u_m^*\nu_m^*)\mathrm{d}\sigma_m \right] \beta(z,\rho) \right\rangle \end{aligned} \tag{21}$$

where $\boldsymbol{\sigma}_m = (\boldsymbol{\rho}_m, k_m), \nu_m = \nu(\boldsymbol{\sigma}_m)$, $u_m = u(z, \boldsymbol{\sigma}_m)$ etc for $m = 1, 2, 3,$. In this we have assumed that moments of all order exist. From equation (12a), $u(z, \boldsymbol{\sigma})$ can be written as

$$u(z, \boldsymbol{\sigma}) = u(0, \boldsymbol{\sigma}) + \int_0^z \left[\frac{\mathrm{i}}{2k}\triangle_\perp^2 u(z', \boldsymbol{\sigma}) + \beta(z', \boldsymbol{\sigma})u\right] \mathrm{d}z'. \tag{22}$$

For a Markov process $u(z, \boldsymbol{\sigma})$ does not depend on $\beta(z', \boldsymbol{\sigma})$ if $z' > z$. If we denote $\triangle z$ as the increment of z, assumed larger than the correlation scale of $\beta(z, \boldsymbol{\sigma})$ in the z direction. We then have

$$u(z, \boldsymbol{\sigma}) = u(z - \triangle z, \boldsymbol{\sigma}) + \int_{z-\triangle z}^z \mathrm{d}z' \left[\frac{\mathrm{i}}{2k}\triangle_\perp^2 u(z', \boldsymbol{\sigma}) + \beta(z', \boldsymbol{\sigma})u(z', \boldsymbol{\sigma})\right]. \tag{23}$$

If $\triangle z$ is small, we can write $u(z, \boldsymbol{\sigma})$ as

$$\begin{aligned} u(z, \boldsymbol{\sigma}) = u(z - \triangle z, \boldsymbol{\sigma}) + \frac{\mathrm{i}}{2k}[\triangle_\perp^2 u(z - \triangle z, \boldsymbol{\sigma})]\triangle z + u(z - \triangle z, \boldsymbol{\sigma}) \\ \times \int_{z-\triangle z}^z \beta(z', \boldsymbol{\sigma})\mathrm{d}z' + \mathrm{o}(\triangle^2 z). \end{aligned} \tag{24}$$

For $\triangle z \to 0$, we have

$$\lim_{\triangle z \to 0} u(z - \triangle z, \boldsymbol{\sigma}) = u(z, \boldsymbol{\sigma}) \tag{25}$$

and

$$\begin{aligned} \left\langle \beta(z, \boldsymbol{\sigma}') \int_{z-\triangle z}^z \beta(z', \boldsymbol{\sigma}) \right\rangle &= \left\langle \frac{\mathrm{i}}{2}k\epsilon_2(z, \boldsymbol{\rho}') \int_{z-\triangle z}^z \left(\frac{\mathrm{i}}{2}k\epsilon_2(z', \boldsymbol{\rho})\right) \mathrm{d}z' \right\rangle \\ &\quad + \left\langle \frac{\mathrm{i}}{2}k\epsilon_3(z, \boldsymbol{\rho}') \int_{z-\triangle z}^z \left(\frac{\mathrm{i}}{2}k\epsilon_3(z', \boldsymbol{\rho})\right) \mathrm{d}z' \right\rangle \\ &= A(\boldsymbol{\sigma} - \boldsymbol{\sigma}') - \tfrac{1}{4}k^2 \left\langle \epsilon_3(\omega, |E|) \int_{z-\triangle z}^z \epsilon_3(\omega, |E|)\mathrm{d}z' \right\rangle. \end{aligned} \tag{26}$$

In deriving the above equation, we have assumed that ϵ_2 and ϵ_3 are statistically independent. If ϵ_3 can be expressed as

$$\epsilon_3 = \epsilon_{30}F(\omega, |u|) \tag{27}$$

where ϵ_{30} is a small quantity and $F(\omega, |u|)$ is an analytical function, we can write equation (26) as

$$\left\langle \beta(z, \boldsymbol{\sigma}') \int_{z-\triangle z}^z \beta(z', \boldsymbol{\sigma}')\mathrm{d}z' \right\rangle = A(\boldsymbol{\sigma} - \boldsymbol{\sigma}') + o(\epsilon_{30}^2) \cong A(\boldsymbol{\sigma} - \boldsymbol{\sigma}') \tag{28}$$

i.e.

$$\langle \beta(z, \boldsymbol{\sigma})\beta(z', \boldsymbol{\sigma}') \rangle = \delta(z - z')A(\boldsymbol{\sigma} - \boldsymbol{\sigma}') \tag{29}$$

and

$$A(\boldsymbol{\sigma}-\boldsymbol{\sigma}') = \frac{1}{2}\int \langle \beta(z,\boldsymbol{\sigma})\beta(z',\boldsymbol{\sigma}')\rangle \mathrm{d}z'. \tag{30}$$

Note that, for convenience and to retain generality, we do not give an explicit form for $A(\boldsymbol{\sigma})$. For higher moments

$$T_i = \left\langle \beta(z,\boldsymbol{\sigma})\int_{z-\Delta z}^{z} \beta(z,\boldsymbol{\sigma}_1)\mathrm{d}\boldsymbol{\sigma}_1 \ldots \beta(z,\boldsymbol{\sigma}_i)\mathrm{d}\boldsymbol{\sigma}_i \right\rangle \qquad i \geq 2. \tag{31}$$

We will assume, as in the usual derivation of the Fokker–Plank equation, that

$$\lim_{\Delta t\to 0} T_i = 0 \qquad i \geq 2. \tag{32}$$

From equations (22), (23), (24), (29) and (31) we have following when $\Delta z \to 0$

$$\langle u(z,\boldsymbol{\sigma})\beta(z,\boldsymbol{\sigma})\rangle = \langle u(z,\boldsymbol{\sigma})\rangle A(\boldsymbol{\sigma}-\boldsymbol{\sigma}') + \tfrac{1}{2}\mathrm{i}k\langle \epsilon_3 u\rangle \tag{33}$$

$$\langle u^*(z,\boldsymbol{\sigma})\beta(z,\boldsymbol{\sigma})\rangle = -\langle u^*(z,\boldsymbol{\sigma})\rangle A(\boldsymbol{\sigma}-\boldsymbol{\sigma}') - \tfrac{1}{2}\mathrm{i}k\langle \epsilon_3 u^*\rangle \tag{34}$$

and in general

$$\begin{aligned}
&\langle (u_1\nu_1 + u_1^*\nu_1^*)\ldots(u_m\nu_m + u_m^*\nu_m^*)\beta(z,\boldsymbol{\sigma})\rangle \\
&\quad = \sum_{j=1}^{m} A(\boldsymbol{\sigma}-\boldsymbol{\sigma}_j)\langle (u_1\nu_1 + u_1^*\nu_1^*)\ldots(u_{j-1}\nu_{j-1} + u_{j-1}^*\nu_{j-1}^*)(u_j\nu_j - u_j^*\nu_j^*) \\
&\qquad \times (u_{j+1}\nu_{j+1} + u_{j+1}^*\nu_{j+1}^*)\ldots(u_m\nu_m + u_m^*\nu_m^*)\rangle \\
&\qquad + \sum_{j=1}^{m} \frac{\mathrm{i}}{2}k_j\langle \epsilon_3(z,\boldsymbol{\sigma}_j)(u_1\nu_1 + u_1^*\nu_1^*)\ldots(u_{j-1}\nu_{j-1} + u_{j-1}^*\nu_{j-1}^*)(u_j\nu_j - u_j^*\nu_j^*) \\
&\qquad \times (u_{j+1}\nu_{j+1} + u_{j+1}^*\nu_{j+1}^*)\ldots(u_m\nu_m + u_m^*\nu_m^*)\rangle.
\end{aligned} \tag{35}$$

By substituting (35) into (21), we have

$$\begin{aligned}
g(\nu,\nu^*,z,\boldsymbol{\sigma}) &= \int A(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\,[\nu\langle u\exp(\mathrm{i}R)\rangle - \nu^*\langle u^*\exp(\mathrm{i}R)\rangle]\,\mathrm{d}\boldsymbol{\sigma}' \\
&\quad + \frac{\mathrm{i}}{2}k\int [\nu\langle \epsilon_3(\omega,|E|)u\exp(\mathrm{i}R)\rangle - \nu^*\langle \epsilon_3(\omega,|E|)u^*\exp(\mathrm{i}R)\rangle]\,\mathrm{d}\boldsymbol{\sigma}'
\end{aligned} \tag{36}$$

and putting equations (17*a*) and (17*b*) into the above, we have

$$\begin{aligned}
g(\nu,\nu^*,z,\boldsymbol{\sigma}) &= -\mathrm{i}\int A(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\left[\nu(\boldsymbol{\sigma})\frac{\delta\Psi}{\delta\nu} - \nu^*(\boldsymbol{\sigma}')\frac{\delta\Psi}{\delta\nu^*}\right]\mathrm{d}\boldsymbol{\sigma}' \\
&\quad + \frac{\mathrm{i}}{2}k\int [\nu\langle \epsilon_3 u\exp(\mathrm{i}R)\rangle - \nu^*\langle \epsilon_3 u^*\exp(\mathrm{i}R)\rangle]\,\mathrm{d}\boldsymbol{\sigma}'.
\end{aligned} \tag{37}$$

We now define an operator $\hat{P}(\boldsymbol{\sigma})$ as follows:

$$\hat{P}(\boldsymbol{\sigma}) = \nu(\boldsymbol{\sigma})\frac{\delta}{\delta\nu} - \nu^*(\boldsymbol{\sigma})\frac{\delta}{\delta\nu^*}. \tag{38}$$

Using this, equation (37) can be written as

$$g(\nu,\nu^*,z,\boldsymbol{\sigma}) = -\mathrm{i}\int A(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\hat{P}\Psi(z,\nu,\nu^*)\mathrm{d}\boldsymbol{\sigma}' + k\int\left[\nu\langle\epsilon_3 u\exp(\mathrm{i}R)\rangle - \nu^*\langle\epsilon_3 u^*\exp(\mathrm{i}R)\rangle\right]\mathrm{d}\boldsymbol{\sigma}'. \tag{39}$$

We also have

$$\langle\beta(z,\boldsymbol{\sigma})u(z,\boldsymbol{\sigma})\exp(\mathrm{i}R)\rangle = \frac{1}{\mathrm{i}}\frac{\delta g(\nu,\nu^*,z,\boldsymbol{\sigma})}{\delta\nu(\boldsymbol{\sigma})} \tag{40}$$

and

$$\langle\beta(z,\boldsymbol{\sigma})u^*(z,\boldsymbol{\sigma})\exp(\mathrm{i}R)\rangle = \frac{1}{\mathrm{i}}\frac{\delta g(\nu,\nu^*,z,\boldsymbol{\sigma})}{\delta\nu^*(\boldsymbol{\sigma})}. \tag{41}$$

Now, using (18*a*), (18*b*) and (37), it is quite straightforward to rewrite (16) as

$$\frac{\delta\Psi}{\delta z} = \frac{\mathrm{i}}{2}\int\mathrm{d}\boldsymbol{\sigma}\frac{1}{k}\left(\nu\Delta_\perp^2\frac{\delta\Psi}{\delta\nu} - \nu^*\Delta_\perp^2\frac{\delta\Psi}{\delta\nu^*}\right) - \int\mathrm{d}\boldsymbol{\sigma}\int\mathrm{d}\boldsymbol{\sigma}' A(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\hat{P}(\boldsymbol{\sigma})\hat{P}(\boldsymbol{\sigma}')\Psi + \frac{\mathrm{i}}{2}\int\mathrm{d}\boldsymbol{\sigma}\int\mathrm{d}\boldsymbol{\sigma}' k\left[\nu\langle\epsilon_3 u\exp(\mathrm{i}R)\rangle - \nu^*\langle\epsilon_3 u^*\exp(\mathrm{i}R)\rangle\right]. \tag{42}$$

This is the differential equation for the characteristic functional Ψ of the random electromagnetic field $u(z,\boldsymbol{\rho})$ in a random and nonlinear medium, where $A(\boldsymbol{\sigma}-\boldsymbol{\sigma}')$ and ϵ_3 depend on the properties of the medium. The first term on the right-hand side of equation (42) is due to diffraction and refraction, the second term on the right-hand side is due to scattering by random fluctuations in the medium and the third term on the right-hand side is due to the nonlinearity of the medium. The differential–integral functional equation (42) includes all the information about the propagation of the wave. There is not, however, a general method of solving such equation directly. We shall approach a solution, therefore, via the hierarchy of coupled moment equations.

2.3. The hierarchy of moment equations

We have derived the differential equation (42) for the characteristic functional of the wavefield. We can now delineate the hierarchy of the moment equations. To achieve this we expand $\Psi(z,\nu,\nu^*)$ as earlier (e.g. (15))

$$\Psi(z,\nu,\nu*) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathrm{i}^{m+n}}{m!n!}\left\langle\left[\int u(z,\boldsymbol{\sigma})\nu(z,\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}\right]^m\left[\int u^*(z,\boldsymbol{\sigma}')\nu^*(z,\boldsymbol{\sigma}')\mathrm{d}\boldsymbol{\sigma}'\right]^n\right\rangle. \tag{43a}$$

This may be written for convenience as

$$\Psi(z,\nu,\nu*) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathrm{i}^{m+n}}{m!n!}M_{mn}(z,\nu,\nu^*) \tag{43b}$$

where

$$M_{mn}(z,\nu,\nu^*) = \int \Gamma_{mn}(z,\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_m,\boldsymbol{\sigma}'_1,\ldots,\boldsymbol{\sigma}'_n)\prod_{j=1}^{m}\nu_j\mathrm{d}\boldsymbol{\sigma}_j\prod_{l=1}^{n}\nu_l^*\mathrm{d}\boldsymbol{\sigma}'_l \tag{44}$$

and

$$\Gamma_{mn}(z,\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_m,\boldsymbol{\sigma}'_1,\ldots,\boldsymbol{\sigma}_{,n}) = \langle u_1,\cdots,u_m u_1^*,\ldots,u_n^*\rangle. \tag{45}$$

For any arbitrary function $f(\boldsymbol{\sigma})$ of $\boldsymbol{\sigma}$, it can readily be shown that the following relations hold:

$$\begin{aligned}\int f(\boldsymbol{\sigma})\nu(\boldsymbol{\sigma})\frac{\delta}{\delta\nu}M_{mn}(z,\nu,\nu^*)\mathrm{d}\boldsymbol{\sigma}\\ = \int\sum_{j=1}^{m}\Gamma_{mn}(z,\boldsymbol{\sigma}_1,\cdot,\boldsymbol{\sigma}_m,\boldsymbol{\sigma}'_1,\cdot,\boldsymbol{\sigma}'_n)f(\boldsymbol{\sigma}_j)\prod_{j=1}^{m}\nu_j\mathrm{d}\boldsymbol{\sigma}_j\prod_{l=1}^{n}\nu_l^*\mathrm{d}\boldsymbol{\sigma}'_l\end{aligned} \tag{46a}$$

and

$$\begin{aligned}\int f(\boldsymbol{\sigma})\nu^*(\boldsymbol{\sigma})\frac{\delta}{\delta\nu^*}M_{mn}(z,\nu,\nu^*)\mathrm{d}\boldsymbol{\sigma}\\ = \int\sum_{j=1}^{m}\Gamma_{mn}(z,\boldsymbol{\sigma}_1,\cdot,\boldsymbol{\sigma}_m,\boldsymbol{\sigma}'_1,\cdot,\boldsymbol{\sigma}'_n)f(\boldsymbol{\sigma}'_j)\prod_{j=1}^{m}\nu_j\mathrm{d}\boldsymbol{\sigma}_j\prod_{l=1}^{n}\nu_l^*\mathrm{d}\boldsymbol{\sigma}'_l.\end{aligned} \tag{46b}$$

So from (44), using (46a), (46b) and (38), we have

$$\begin{aligned}\int\int A(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\hat{P}(\boldsymbol{\sigma})\hat{P}(\boldsymbol{\sigma}')M_{mn}(z,\nu,\nu^*)\mathrm{d}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{\sigma}'\\ = \int\Bigg[\sum_{j=1}^{m}\sum_{l=1}^{m}A(\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_l) - \sum_{j=1}^{m}\sum_{l=1}^{n}[A(\boldsymbol{\sigma}_j-\boldsymbol{\sigma}'_l)+A(\boldsymbol{\sigma}'_l-\boldsymbol{\sigma}_j)]\\ +\sum_{j=1}^{n}\sum_{l=1}^{n}A(\boldsymbol{\sigma}'_j-\boldsymbol{\sigma}'_l)\Bigg]\Gamma_{mn}\prod_{j=1}^{m}\nu_j\mathrm{d}\boldsymbol{\sigma}_j\prod_{l=1}^{m}\nu_l^*\mathrm{d}\boldsymbol{\sigma}'_l.\end{aligned} \tag{47}$$

We also have

$$\int\nu(\boldsymbol{\sigma})\Delta_\perp^2\frac{\delta M_{mn}}{\delta\nu}\mathrm{d}\boldsymbol{\sigma} = \int(\Delta_1^2+\ldots+\Delta_m^2)\Gamma_{mn}\prod_{j=1}^{m}\nu_j\mathrm{d}\boldsymbol{\sigma}_j\prod_{l=1}^{n}\nu_l^*\mathrm{d}\boldsymbol{\sigma}'_l \tag{48a}$$

and

$$\int\nu^*(\boldsymbol{\sigma})\Delta_\perp^2\frac{\delta M_{mn}}{\delta\nu^*}\mathrm{d}\boldsymbol{\sigma} = \int(\Delta_1'^2+\ldots+\Delta_m'^2)\Gamma_{mn}\prod_{j=1}^{m}\nu_j\mathrm{d}\boldsymbol{\sigma}_j\prod_{l=1}^{n}\nu_l^*\mathrm{d}\boldsymbol{\sigma}'_l. \tag{48b}$$

Using equations (43), (47), (48a) and (48b) the differential–integral functional equation (42) can be written as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^{m+n}}{m!n!}\Bigg\{\int\Bigg[\frac{\partial}{\partial z}-\frac{\mathrm{i}}{2}\left(\sum_{j=1}^{m}\frac{\Delta_j^2}{k_j}-\sum_{l=1}^{n}\frac{\Delta_l'^2}{k'_l}\right)+\sum_{j=1}^{m}\sum_{l=1}^{m}A(\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_l)$$

$$- \sum_{j=1}^{m}\sum_{l=1}^{n}[A(\sigma_j - \sigma'_l) + A(\sigma'_l - \sigma_j)] + \sum_{j=1}^{n}\sum_{l=1}^{n} A(\sigma'_j - \sigma'_l)\Bigg]\Gamma_{mn}$$

$$\times \prod_{j=1}^{m} \nu_j \mathrm{d}\sigma_j \prod_{l=1}^{n} \nu_l^* \mathrm{d}\sigma'_l$$

$$- \frac{\mathrm{i}}{2}\int\left[\sum_{j=1}^{m} k_j \langle \epsilon_3 \prod_{j=1}^{m} u_j \prod_{l=1}^{n} u_l^* \rangle - \sum_{l=1}^{n} k'_l \langle \epsilon_3 \prod_{j=1}^{m} u_j \prod_{l=1}^{n} u_l^* \rangle\right]$$

$$\times \prod_{j=1}^{m} \nu_j \mathrm{d}\sigma_j \prod_{l=1}^{n} \nu_l^* \mathrm{d}\sigma_l \Bigg\} = 0. \tag{49}$$

Since $\nu(\sigma)$, $\nu^*(\sigma')$ are arbitrary, the quantity inside the curly brackets in equation (49) must be zero. Therefore we get the following differential–integral equation for the moment of the wavefield:

$$\Bigg\{\frac{\partial}{\partial z} - \frac{\mathrm{i}}{2}\left(\sum_{j=1}^{m}\frac{\Delta_j^2}{k_j} - \sum_{l=1}^{n}\frac{\Delta_l'^2}{k'_l}\right) + \sum_{j=1}^{m}\sum_{l=1}^{m} A(\sigma_j - \sigma_l)$$

$$- \sum_{j=1}^{m}\sum_{l=1}^{n}[A(\sigma_j - \sigma'_l) + A(\sigma'_l - \sigma_j)] + \sum_{j=1}^{n}\sum_{l=1}^{n} A(\sigma'_j - \sigma'_l)\Bigg\}\Gamma_{mn}$$

$$- \frac{\mathrm{i}}{2}\left[\sum_{j=1}^{m} k_j \langle \epsilon_3(\sigma_j) \prod_{j=1}^{m} u_j \prod_{l=1}^{n} u_l^* \rangle - \sum_{l=1}^{n} k'_l \langle \epsilon_3(\sigma_l) \prod_{j=1}^{m} u_j \prod_{l=1}^{n} u_l^* \rangle\right] = 0. \tag{50}$$

For convenience we shall write

$$F_{mn} = \sum_{j=1}^{m}\sum_{l=1}^{m} A(\sigma_j - \sigma_l) - \sum_{j=1}^{m}\sum_{l=1}^{n}[A(\sigma_j - \sigma'_l) + A(\sigma'_l - \sigma_j)] + \sum_{j=1}^{n}\sum_{l=1}^{n} A(\sigma'_j - \sigma'_l). \tag{51}$$

We note that from (50) we recover equation (19*a*) in [15], if the nonlinearity is specified in the same way as in [15]. As we have pointed out earlier, the wave will be modulated in space and time when it propagates in a turbulent and nonlinear medium, creating new components with new frequencies and new wavevectors. It is desirable therefore to know the moments of the wavefield as a function of wavenumber and the position. To this end we can derive an hierarchy of coupled partial differential equations containing different wavenumbers. If the nonlinearity is as follows

$$\epsilon_3 = \epsilon_{30}|E|^2 \tag{51b}$$

where ϵ_{30} is a small constant, E, the total wavefield resulting from the full range of wavenumbers, can now be more generally expressed as

$$E = \sum_{p=1}^{P} u_p \exp(\mathrm{i}k_p z)$$

where u_p are all the components of the wave with different wavenumbers. P is the number of all possible components of the wave. We then have, for the case (51*b*), the following hierarchy of coupled partial differential equations for the moments:

$$\frac{\partial}{\partial z}\Gamma_{mn} = \frac{\mathrm{i}}{2}\left[\sum_{j=1}^{m}\frac{\Delta_j^2}{k_j} - \sum_{l=1}^{n}\frac{\Delta_l'^2}{k_l'}\right]\Gamma_{mn} - F_{mn}\Gamma_{mn} - \frac{\mathrm{i}}{2}\epsilon_{30}\left[\sum_{j=1}^{m}k_j\left(\sum_{p,q=1}^{P}\Gamma_{(m+1)(n+1)}(p_j,q_j)\right) - \sum_{l=1}^{n}k_l'\left(\sum_{p,q=1}^{P}\Gamma_{(m+1)(n+1)}(p_l,q_l)\right)\right] \tag{52}$$

where

$$\Gamma_{(m+1)(n+1)}(p_i,q_i) = \left\langle u_p(z,\rho_i)u_q^*(z,\rho_i)\prod_{j=1}^{m}u_j(z,\rho_j)\prod_{l=1}^{n}u_l^*(z,\rho_l)\right\rangle .$$

When, on the other hand, the nonlinearity of the random medium has an exponential form, as described by equation (14), for $\alpha = 1$

$$\epsilon_3(\omega,|E|) = \epsilon_{30}[1-\exp(-|E|^2)] \tag{14}$$

we arrive at the following hierarchy:

$$\frac{\partial}{\partial z}\Gamma_{mn} = \frac{\mathrm{i}}{2}\left[\sum_{j=1}^{m}\frac{\Delta_j^2}{k_j} - \sum_{l=1}^{n}\frac{\Delta_l'^2}{k_l'}\right]\Gamma_{mn} - F_{mn}\Gamma_{mn} - \frac{\mathrm{i}}{2}\epsilon_{30}\left[\sum_{j=1}^{m}k_j\left(\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r!}\sum_{p,q=1}^{P}\Gamma_{(m+r)(n+r)}(p_j,q_j)\right) - \sum_{l=1}^{n}k_l'\left(\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r!}\sum_{p,q=1}^{P}\Gamma_{(m+r)(n+r)}(p_l,q_l)\right)\right] \tag{53}$$

where

$$\Gamma_{(m+r)(n+r)}(p_i,q_i) = \left\langle u_p^r(z,\rho_i)u_q^{*r}(z,\rho_i)\prod_{j=1}^{m}u_j(z,\rho_j)\prod_{l=1}^{n}u_l^*(z,\rho_l)\right\rangle .$$

It is clear that all the sets of equations (52) and (53) form chains. They cannot be solved for Γ_{mn} as they stand, since $\Gamma_{(m+1)(n+1)}$, $\Gamma_{(m+2)(n+2)}$ etc are unknown. Additional information is required before the chains can be established. We shall discuss this in the following section.

3. An approximate method for solving the hierarchy of the moment equations

There is at present no exact solution for the hierarchy of the moment equations. However, an approximate solution can be found if additional information can be introduced, as we now demonstrate by showing how equation (52) can be solved using a perturbation method.

In deriving the hierarchies (52) and (53), we have assumed that the nonlinearity is weak. In which case we treat ϵ_{30}, which $\epsilon_{30} << \epsilon_0$, as a small parameter, and assume that Γ_{mn} can be considered as perturbation series, i.e.

$$\begin{aligned}\Gamma_{mn} &= \Gamma^{(0)}_{mn} + \epsilon_{30}\Gamma^{(1)}_{mn} + \epsilon_{30}^2\Gamma^{(2)}_{mn} + \cdot \\ &= \sum_{r=0}^{\infty} \epsilon_{30}^r \Gamma^{(r)}_{mn}\end{aligned} \tag{54}$$

where $\Gamma^{(r)}_{mn}$ is the rth-order term of the wave moment. By substituting (54) into (52) and equating the terms containing the same order of ϵ_{30}, we have

$$\frac{\partial}{\partial z}\Gamma^{(0)}_{mn} = \frac{\mathrm{i}}{2}\left[\sum_{j=1}^{m}\frac{\Delta_j^2}{k_j} - \sum_{l=1}^{n}\frac{\Delta_l'^2}{k_l'}\right]\Gamma^{(0)}_{mn} - F_{mn}\Gamma^{(0)}_{mn} \tag{55}$$

$$\cdots$$

$$\begin{aligned}\frac{\partial}{\partial z}\Gamma^{(r)}_{mn} = \frac{\mathrm{i}}{2}&\left[\sum_{j=1}^{m}\frac{\Delta_j^2}{k_j} - \sum_{l=1}^{n}\frac{\Delta_l'^2}{k_l'}\right]\Gamma^{(r)}_{mn} - F_{mn}\Gamma^{(r)}_{mn} \\ &- \frac{\mathrm{i}}{2}\left[\sum_{j=1}^{m} k_j\left(\sum_{p,q=1}^{P}\Gamma^{(r-1)}_{(m+1)(n+1)}(p_j, q_j)\right)\right. \\ &\left. - \sum_{l=1}^{n} k_l'\left(\sum_{p,q=1}^{P}\Gamma^{(r-1)}_{(m+1)(n+1)}(p_l, q_l)\right)\right] \qquad \text{for } r \geq 1.\end{aligned} \tag{56}$$

We recognise $\Gamma^{(0)}_{mn}$ as the linear part of Γ_{mn}, and $\Gamma^{(r)}_{mn}$ are all the nonlinear parts for $r \geq 1$. The hierarchy of coupled partial differential equations has then been decoupled, allowing the solution to be found. Using the solutions found earlier for equation (55) [7–9], we have the moment of wave at a point

$$\Gamma^{(0)}_{mn}(z, \rho_1, \cdot, \rho_m, \rho_1', \cdot, \rho_n') = \sum_{p=0}^{\infty}\Gamma'^{(p)}_{mn}(z, \rho_1, \cdot, \rho_m, \rho_1', \cdot, \rho_n') \tag{57}$$

where

$$\begin{aligned}\Gamma'^{(0)}_{mn}(z, \rho_1, \cdot, \rho_m, \rho_1', \cdot, \rho_n') &= (-1)^m\left(\frac{\mathrm{i}}{2\pi z}\right)^{m+n} \\ &\times \prod_{j=1}^{m} k_j \prod_{l=1}^{n} k_l' \int_{-\infty}^{\infty}\Gamma_{mn}(0, \tilde{\rho}_1, \cdot, \tilde{\rho}_m, \tilde{\rho}_1', \cdot, \tilde{\rho}_n') \\ &\times \exp\left(\sum_{j=1}^{m}\mathrm{i}k_j\frac{(\rho_j - \tilde{\rho}_j)^2}{2z} - \sum_{l=1}^{n}\mathrm{i}k_l'\frac{(\rho_l' - \tilde{\rho}_l')^2}{2z}\right)\prod_{j=1}^{m}\mathrm{d}\tilde{\rho}_j\prod_{l=1}^{n}\mathrm{d}\tilde{\rho}_l'\end{aligned} \tag{58}$$

and

$$\Gamma'^{(p)}_{mn}(z,\rho_1,\cdot,\rho_m,\rho'_1,\cdot,\rho'_n) = -(-1)^m\left(\frac{\mathrm{i}}{2\pi}\right)^{m+n}\prod_{j=1}^{m}k_j\prod_{l=1}^{n}k'_l\int_0^z \mathrm{d}z'\int_{-\infty}^{\infty}\left(\frac{1}{z-z'}\right)^{m+n}$$
$$\times F_{mn}(\tilde{\rho}_1,\cdot,\tilde{\rho}_m,\tilde{\rho}'_1,\cdot,\tilde{\rho}'_n)\Gamma'^{(p-1)}_{mn}(z',\tilde{\rho}_1,\cdot,\tilde{\rho}_m,\tilde{\rho}'_1,\cdot,\tilde{\rho}'_n)$$
$$\times\exp\left(\sum_{j=1}^{m}\mathrm{i}k_j\frac{(\rho_j-\tilde{\rho}_j)^2}{2(z-z')}-\sum_{l=1}^{n}\mathrm{i}k'_l\frac{(\rho'_l-\tilde{\rho}'_l)^2}{2(z-z')}\right)\prod_{j=1}^{m}d\tilde{\rho}_j\prod_{l=1}^{n}d\tilde{\rho}'_l$$
$$\text{for } p\geq 1 \tag{59}$$

and from the initial condition

$$\Gamma_{mn}(0,\tilde{\rho}_1,\cdot,\tilde{\rho}_m,\tilde{\rho}'_1,\cdot,\tilde{\rho}'_n) = \prod_{j=1}^{m}u(0,\tilde{\rho}_j)\prod_{l=1}^{n}u^*(0,\tilde{\rho}'_l). \tag{60}$$

Then the solution of equation (56) is as follows:

$$\Gamma^{(r)}_{mn}(z,\rho_1,\cdot,\rho_m,\rho'_1,\cdot,\rho'_n) = \sum_{p=0}^{\infty}\Gamma^{(r,p)}_{mn}(z,\rho_1,\cdot,\rho_m,\rho'_1,\cdot,\rho'_n)\qquad r\geq 1 \tag{61}$$

where

$$\begin{aligned}\Gamma^{(r,0)}_{mn}&(z,\rho_1,\cdot,\rho_m,\rho'_1,\cdot,\rho'_n)\\
={}& -(-1)^m\left(\frac{\mathrm{i}}{2\pi}\right)^{m+n}\prod_{j=1}^{m}k_j\prod_{l=1}^{n}k'_l\int_0^z \mathrm{d}z'\int_{-\infty}^{\infty}\left(\frac{1}{z-z'}\right)^{m+n}\\
&\times\left\{-\frac{\mathrm{i}}{2}\left[\sum_{j=1}^{m}k_j\left(\sum_{p,q=1}^{P}\Gamma^{(r-1)}_{(m+1)(n+1)}(\tilde{\rho}_j,\tilde{q}_j)\right)\right.\right.\\
&\left.\left.-\sum_{l=1}^{n}k'_l\left(\sum_{p,q=1}^{P}\Gamma^{(r-1)}_{(m+1)(n+1)}(\tilde{\rho}_l,\tilde{q}_l)\right)\right]\right\}\\
&\times\exp\left(\sum_{j=1}^{m}\mathrm{i}k_j\frac{(\rho_j-\tilde{\rho}_j)^2}{2(z-z')}-\sum_{l=1}^{n}\mathrm{i}k'_l\frac{(\rho'_l-\tilde{\rho}'_l)^2}{2(z-z')}\right)\prod_{j=1}^{m}d\tilde{\rho}_j\prod_{l=1}^{n}d\tilde{\rho}'_l\end{aligned} \tag{62}$$

and

$$\begin{aligned}\Gamma^{(r,p)}_{mn}&(z,\rho_1,\cdot,\rho_m,\rho'_1,\cdot,\rho'_n)\\
={}& -(-1)^m\left(\frac{\mathrm{i}}{2\pi}\right)^{m+n}\prod_{j=1}^{m}k_j\prod_{l=1}^{n}k'_l\int_0^z \mathrm{d}z'\int_{-\infty}^{\infty}\left(\frac{1}{z-z'}\right)^{m+n}\\
&\times F_{mn}(\tilde{\rho}_1,\cdot,\tilde{\rho}_m,\tilde{\rho}'_1,\cdot,\tilde{\rho}'_n)\Gamma^{(r,p-1)}_{mn}(z',\tilde{\rho}_1,\cdot,\tilde{\rho}_m,\tilde{\rho}'_1,\cdot,\tilde{\rho}'_n)\\
&\times\exp\left(\sum_{j=1}^{m}\mathrm{i}k_j\frac{(\rho_j-\tilde{\rho}_j)^2}{2(z-z')}-\sum_{l=1}^{n}\mathrm{i}k'_l\frac{(\rho'_l-\tilde{\rho}'_l)^2}{2(z-z')}\right)\prod_{j=1}^{m}\mathrm{d}\tilde{\rho}_j\prod_{l=1}^{n}\mathrm{d}\tilde{\rho}'_l\\
&\text{for } p\geq 1\end{aligned} \tag{63}$$

and the $\Gamma^{(r-1)}_{(m+1)(n+1)}(\tilde{p}_l,\tilde{q}_l)$ is defined as in (52) but at positions having two-dimensional radii as $\tilde{\rho}_j$, $\tilde{\rho}'_l$ etc. Therefore we can obtain the solutions for equations (55), (56) by starting with the solution of the linear m-nth moment equation to get the high order terms of the solution.

4. Summary and discussion

We have derived the functional equation (42) for the characteristic functional of the wavefield, for an initially monochromatic wave propagating through a turbulent and nonlinear plasma. It conveys all the information about the wave. However, there are at present no general methods of solving equations in functional derivatives. Therefore we have developed a technique to derive a hierarchy of equations for the moments of the wavefield. It is general and does not assume any particular form of the random distribution. We have examined two cases of nonlinearity; one quadratic and one exponential for $\epsilon_0(E)$. It is well known that the quadratic nonlinearity for ϵ_0, leading to cubic nonlinear equation, is equivalent to the four-wave resonant interaction process; and the exponential nonlinearity includes all wave resonant interactions [16]. The hierarchy of moment equations was solved using a perturbation method.

It is worth noting that the structure of the hierarchy of the coupled moment equations of a wave in a turbulent and nonlinear medium is similar to the structure of the hierarchy of the Vlasov cumulent for weak turbulence [20], and to the BBGKY hierarchy of equations in statistical mechanics [21]. This actually shows that there are features shared by different stochastic processes. This point will be pursued in a future work.

There are various applications. The first moment of the wave describes the average evolution of a nonlinear wave propagating in a turbulent plasma. The second moment of the wave at the same point yields the spatial variation in intensity. The fourth moment could be used to study scintillation in a turbulent and nonlinear plasma. The moments can also be contructed to study pulse broadening in time and in space, and time delays between signals, etc [8].

We conclude with an example outlining one way in which our results could be used. For this we consider the important problem of long distance propagation of laser radiation in turbulent space plasmas. It is well known that when a strong enough electromagnetic wave propagates in a plasma, it produces density perturbations and excites low-frequency waves, the Stokes electromagnetic wave and the anti-Stokes electromagnetic wave [22]. The nonlinear process is one of four-wave nonlinear coupling and can be described by the cubic nonlinear equation. For this we have to assess the combined effects of plasma turbulence and the nonlinear interaction of the light with the plasma. The spatial broadening of a beam due to turbulence can reduce the self-induced effects of waves; and, in turn, nonlinearities in the medium can alter the nature of the interaction between the beam and turbulent inhomogeneities, as a result both of the formation of a wave-guiding channel in the medium and due to the randomisation of the medium in the field of a randomly modulated large-amplitude wave. To evaluate the broadening of a beam, we need to evaluate the mean square beam radius $\langle\rho^2\rangle$, defined as [23]:

$$\langle\rho^2\rangle = \frac{\int_{-\infty}^{\infty}\rho^2\Gamma_{11}(z,\rho)\mathrm{d}\rho}{\int_{-\infty}^{\infty}\Gamma_{11}(z,\rho)\mathrm{d}\rho}. \tag{64}$$

Once Γ_{11} is known we can readily calculate $\langle\rho^2\rangle$. We shall assume that the initial field distribution has a Gaussian form

$$u_0(\rho) = \exp\left[-\frac{2\rho^2}{D^2} - \frac{\mathrm{i}k\rho^2}{2F}\right] \tag{65}$$

representing a beam with an initial characteristic diameter D and radius of curvature F. For this distribution the initial $\Gamma_{11}(z=0,\rho,\rho')$ is, from (60)

$$\Gamma_{11}(z=0,\rho,\rho') = \exp\left[-\frac{2(\rho^2+\rho'^2)}{D^2}\right]. \tag{66}$$

If the refractive index irregularities in the medium are isotropic and have a Gaussian autocorrelation function, the transverse autocorrelation function is

$$A(\sigma_i - \sigma_j') = \frac{1}{k_i k_j'} \exp\left[-\frac{(\rho_i - \rho_j')^2}{L^2}\right] \tag{67}$$

where L is the characteristic length of the turbulence.

Therefore we have

$$F_{11}(\rho_1,\rho_1') = \frac{1}{k_1^2} - \frac{2}{k_1 k_1'}\exp\left[-\frac{(\rho_1-\rho_1')^2}{L^2}\right] + \frac{1}{k_1'^2}. \tag{68}$$

The second moment is composed of the parts due to linear effects and nonlinear effects. The part due to linear effects is given by (57). It can be written as

$$\Gamma_{11}^{(0)}(z,\rho) = \sum_{p=0}^{\infty} \Gamma_{11}'^{(p)}(z,\rho). \tag{69}$$

By putting (66) in (58) and (68) in (59), we have

$$\begin{aligned}\Gamma_{11}'^{(0)}(z,\rho) = -\left(\frac{\mathrm{i}}{2\pi z}\right)^2 k_1 k_1' \int_{-\infty}^{\infty} \exp\left[-\frac{2(\rho_1^2+\rho_1'^2)}{D^2}\right] \\ \times \exp\left(\mathrm{i}k_1\frac{(\rho-\rho_1)^2}{2z} - \mathrm{i}k_1'\frac{(\rho-\rho_1')^2}{2z}\right)\mathrm{d}\rho_1 d\rho_1'\end{aligned} \tag{70}$$

and

$$\begin{aligned}\Gamma_{11}'^{(p)}(z,\rho) = -\left(\frac{\mathrm{i}}{2\pi}\right)^2 k_1 k_1' \int_0^z \mathrm{d}z' \int_{-\infty}^{\infty} \left(\frac{1}{z-z'}\right)^2 \\ \times\left\{\frac{1}{k_1^2} - \frac{2}{k_1 k_1'}\exp\left[-\frac{(\rho_1-\rho_1')^2}{L^2}\right] + \frac{1}{k_1'^2}\right\}\Gamma_{11}'^{(p-1)}(z',\rho_1,\rho_1') \\ \times \exp\left(\mathrm{i}k_1\frac{(\rho-\rho_1)^2}{2(z-z')} - \mathrm{i}k_1'\frac{(\rho-\rho_1')^2}{2(z-z')}\right)\mathrm{d}\rho_1\mathrm{d}\rho_1' \qquad p \geq 1.\end{aligned} \tag{71}$$

The part of the second moment due to nonlinear effects can be obtained from (61). For this example we neglect moments higher than fourth order, cutting off the chain there. The first-order nonlinear contribution to the second moment can be written, from (61), as

$$\Gamma_{11}^{(1)}(z,\rho) = \sum_{p=0}^{\infty} \Gamma_{11}^{(1,p)}(z,\rho) \tag{72}$$

where, following (62)

$$\Gamma_{11}^{(1,0)}(z,\rho) = -\left(\frac{\mathrm{i}}{2\pi}\right)^2 k_1 k_1' \int_0^z \mathrm{d}z' \int_{-\infty}^{\infty} \left(\frac{1}{z-z'}\right)^2 \times \left\{-\frac{\mathrm{i}}{2}\left[k_1\left(\sum_{p,q=1}^{4} \Gamma_{22}^{(0)}(z',\rho_1,\rho_1,\rho_1,\rho_1')\right) - k_1'\left(\sum_{p,q=1}^{4} \Gamma_{22}^{(0)}(z',\rho_1,\rho_1',\rho_1',\rho_1')\right)\right]\right\} \times \exp\left(\mathrm{i}k_1\frac{(\rho-\rho_1)^2}{2(z-z')} - \mathrm{i}k_1'\frac{(\rho-\rho_1')^2}{2(z-z')}\right)\mathrm{d}\rho_1\mathrm{d}\rho_1' \tag{73}$$

and, following (63)

$$\Gamma_{11}^{(1,p)}(z,\rho) = -\left(\frac{\mathrm{i}}{2\pi}\right)^2 k_1 k_1' \int_0^z \mathrm{d}z' \int_{-\infty}^{\infty} \left(\frac{1}{z-z'}\right)^2 \times \left\{\frac{1}{k_1^2} - \frac{2}{k_1 k_1'}\exp\left[-\frac{(\rho_1-\rho_1')^2}{L^2}\right] + \frac{1}{k_1'^2}\right\}\Gamma_{11}^{(1,p-1)}(z',\rho_1,\rho_1') \times \exp\left(\mathrm{i}k_1\frac{(\rho-\rho_1)^2}{2(z-z')} - \mathrm{i}k_1'\frac{(\rho-\rho_1')^2}{2(z-z')}\right)d\rho_1\mathrm{d}\rho_1' \qquad p \geq 1. \tag{74}$$

Similarly, $\Gamma_{22}^{(0)}(p,q)$ in (73) can be found using (57), (58) and (59), to be

$$\Gamma_{22}^{(0)}(z,\rho_1,\rho_2,\rho_1',\rho_2') = \sum_{p=0}^{\infty} \Gamma_{22}'^{(p)}(z,\rho_1,\rho_2,\rho_1',\rho_2') \tag{75}$$

where

$$\Gamma_{22}'^{(0)}(z,\rho_1,\rho_2,\rho_1',\rho_2') = \left(\frac{\mathrm{i}}{2\pi z}\right)^4 k_1 k_2 k_1' k_2' \int_{-\infty}^{\infty} \Gamma_{22}(0,\tilde{\rho}_1,\tilde{\rho}_2,\tilde{\rho}_1',\tilde{\rho}_2') \times \exp\left(\sum_{j=1}^{2}\mathrm{i}k_j\frac{(\rho_j-\tilde{\rho}_j)^2}{2z} - \sum_{l=1}^{2}\mathrm{i}k_l'\frac{(\rho_l'-\tilde{\rho}_l')^2}{2z}\right)d\tilde{\rho}_1\mathrm{d}\tilde{\rho}_2\mathrm{d}\tilde{\rho}_1'\mathrm{d}\tilde{\rho}_2' \tag{76}$$

and

$$\Gamma_{22}'^{(p)}(z,\rho_1,\rho_2,\rho_1',\rho_2') = -\left(\frac{\mathrm{i}}{2\pi}\right)^4 k_1 k_2 k_1' k_2' \int_0^z \mathrm{d}z' \int_{-\infty}^{\infty} \left(\frac{1}{z-z'}\right)^4 \times F_{22}(\tilde{\rho}_1,\tilde{\rho}_2,\tilde{\rho}_1',\tilde{\rho}_2')\Gamma_{22}'^{(p-1)}(z',\tilde{\rho}_1,\tilde{\rho}_2,\tilde{\rho}_1',\tilde{\rho}_2') \times \exp\left(\sum_{j=1}^{2}\mathrm{i}k_j\frac{(\rho_j-\tilde{\rho}_j)^2}{2(z-z')} - \sum_{l=1}^{2}\mathrm{i}k_l'\frac{(\rho_l'-\tilde{\rho}_l')^2}{2(z-z')}\right)\mathrm{d}\tilde{\rho}_1\mathrm{d}\tilde{\rho}_2\mathrm{d}\tilde{\rho}_1'\mathrm{d}\tilde{\rho}_2' \quad \text{for } p \geq 1. \tag{77}$$

The initial Γ_{22} and F_{22}, from (60) and (51), are as follows:

$$\Gamma_{22}(0,\tilde{\rho}_1,\tilde{\rho}_2,\tilde{\rho}_1',\tilde{\rho}_2') = \exp\left[-\frac{2(\tilde{\rho}_1^2+\tilde{\rho}_2^2+\tilde{\rho}_1'^2+\tilde{\rho}_2'^2)}{D^2}\right] \tag{78}$$

and

$$F_{22}(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_1', \tilde{\rho}_2')$$
$$= \sum_{j=1}^{2}\sum_{l=1}^{2} \frac{1}{k_j k_l} \exp\left[-\frac{(\tilde{\rho}_j - \tilde{\rho}_l)^2}{L^2}\right] - \sum_{j=1}^{2}\sum_{l=1}^{2}\left\{\frac{1}{k_j k_l'} \exp\left[-\frac{(\tilde{\rho}_j - \tilde{\rho}_l')^2}{L^2}\right]\right.$$
$$\left. + \frac{1}{k_j k_l'} \exp\left[-\frac{(\tilde{\rho}_l' - \tilde{\rho}_j)^2}{L^2}\right]\right\} + \sum_{j=1}^{2}\sum_{l=1}^{2} \frac{1}{k_j' k_l'} \exp\left[-\frac{(\tilde{\rho}_j' - \tilde{\rho}_l')^2}{L^2}\right]. \quad (79)$$

Having thus specified F_{11}, F_{22}, $\Gamma_{11}(0, \tilde{\rho}_1, \tilde{\rho}_1')$, $\Gamma_{22}(0, \tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_1', \tilde{\rho}_2')$ for a particular problem we have demonstrated the method of finding the beam radius $\langle \rho^2 \rangle$ from (64) for a wave of interest. For the numerical calculation of the high dimensional integrals, it is necessary to use some results of modern number theory. Such computations are currently under consideration.

Acknowledgments

One of us, Zhen-song Wang, is most grateful for financial support from the K C Wong Education Foundation. He also thanks the Royal Society of London for his Royal Society K C Wong fellowship and the Rutherford Appleton Laboratory for their hospitality.

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